7.1

INTRODUCTION TO PERIODIC FUNCTIONS
Ferris Wheel Height As a Function of Time

The London Eye Ferris Wheel measures 450 feet in diameter and turns continuously, completing a single rotation once every 30 minutes. Suppose you hop on the London Eye Ferris wheel at time $t = 0$ and ride it for two full turns. Let $f(t)$ be your height above the ground, measured in feet as a function of $t$, the number of minutes you have been riding. We can figure out some values of $f(t)$. Since the speed of the rotation is constant, you are at the top 15 minutes after boarding and one-quarter of the way up at 7.5 minutes and 22.5 minutes after boarding. Then you are back at the bottom after 30 minutes and the process continues.

Values of $f(t)$, your height above the ground $t$ minutes after boarding the wheel

<table>
<thead>
<tr>
<th>$t$ (minutes)</th>
<th>0</th>
<th>7.5</th>
<th>15</th>
<th>22.5</th>
<th>30</th>
<th>37.5</th>
<th>45</th>
<th>52.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(t)$ (feet)</td>
<td>0</td>
<td>225</td>
<td>450</td>
<td>225</td>
<td>0</td>
<td>225</td>
<td>450</td>
<td>225</td>
</tr>
<tr>
<td>$t$ (minutes)</td>
<td>60</td>
<td>67.5</td>
<td>75</td>
<td>82.5</td>
<td>90</td>
<td>97.5</td>
<td>105</td>
<td>112.5</td>
</tr>
<tr>
<td>$f(t)$ (feet)</td>
<td>0</td>
<td>225</td>
<td>450</td>
<td>225</td>
<td>0</td>
<td>225</td>
<td>450</td>
<td>225</td>
</tr>
</tbody>
</table>
Graphing the Ferris Wheel Function

Values of $f(t)$, your height above the ground $t$ minutes after boarding the wheel

<table>
<thead>
<tr>
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<th>15</th>
<th>22.5</th>
<th>30</th>
<th>37.5</th>
<th>45</th>
<th>52.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(t)$ (feet)</td>
<td>0</td>
<td>225</td>
<td>450</td>
<td>225</td>
<td>0</td>
<td>225</td>
<td>450</td>
<td>225</td>
</tr>
<tr>
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<td>60</td>
<td>67.5</td>
<td>75</td>
<td>82.5</td>
<td>90</td>
<td>97.5</td>
<td>105</td>
<td>120</td>
</tr>
<tr>
<td>$f(t)$ (feet)</td>
<td>0</td>
<td>225</td>
<td>450</td>
<td>225</td>
<td>0</td>
<td>225</td>
<td>450</td>
<td>225</td>
</tr>
</tbody>
</table>

Notice that the values of $f(t)$ in the table begin repeating after 30 minutes. This is because the second turn is just like the first turn, except that it happens 30 minutes later. If you ride the wheel for more full turns, the values of $f(t)$ continue to repeat at 30-minute intervals. We plot this data and fill in the blank spaces.
The Ferris Wheel function, $f$, is said to be periodic, because its values repeat on a regular interval or period. In the figure, the period is indicated by the horizontal gap between the first two peaks. The dashed horizontal line is the midline of the graph of $f$. The vertical distance shown between the first peak and the midline is called the amplitude.
THE SINE AND COSINE FUNCTIONS
Using Angles to Measure Position On a Circle

Conventions For Working With Angles

• We measure angles with respect to the horizontal, not the vertical, so that 0° describes the 3 o’clock position.

• Positive angles are measured in the counter-clockwise direction, negative angles in the clockwise direction.

• Large angles (greater than 360° or less than −360°) wrap around a circle more than once.
Height on the Ferris Wheel as a Function of Angle

Since we can measure position on the Ferris wheel using angles, we see that the height above the ground is a function of the angle position on the wheel. We can rewrite our table giving heights as a function of angle, instead of time.

*Your height above ground, y, as a function of the angle turned through by the wheel*

<table>
<thead>
<tr>
<th>θ (degrees)</th>
<th>-90°</th>
<th>0°</th>
<th>90°</th>
<th>180°</th>
<th>270°</th>
<th>360°</th>
<th>450°</th>
<th>540°</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(t) (feet)</td>
<td>0</td>
<td>225</td>
<td>450</td>
<td>225</td>
<td>0</td>
<td>225</td>
<td>450</td>
<td>225</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>θ (degrees)</th>
<th>630°</th>
<th>720°</th>
<th>810°</th>
<th>900°</th>
<th>990°</th>
<th>1080°</th>
<th>1170°</th>
<th>1260°</th>
<th>1350°</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(t) (feet)</td>
<td>0</td>
<td>225</td>
<td>450</td>
<td>225</td>
<td>0</td>
<td>225</td>
<td>450</td>
<td>225</td>
<td>0</td>
</tr>
</tbody>
</table>

Recall how the y-values repeat every 30 minutes. Similarly, in the Table, the values of y repeat every 360°. In both cases, the y-values repeat every time the wheel completes one full revolution.
The Unit Circle

When we studied quadratic functions, we transformed a special “starting” function \( y = x^2 \) to get other quadratic functions. Similarly, we begin here with a special “starting” circle. This is the unit circle, the circle of radius one centered at the origin. The unit circle gets its name from the fact that its radius measures exactly one unit.
The Sine and Cosine Functions

Suppose $P = (x, y)$ in the figure is the point on the unit circle specified by the angle $\theta$. We define the functions, cosine of $\theta$, or $\cos \theta$, and sine of $\theta$, or $\sin \theta$, by

$$\cos \theta = x \text{ and } \sin \theta = y.$$ 

In other words, $\cos \theta$ is the $x$-coordinate of the point $P$; and $\sin \theta$ is the $y$-coordinate.
Values of the Sine and Cosine Functions

Example 2
Find the values of $\sin \theta$ and $\cos \theta$ for $\theta = 0^\circ, 90^\circ, 180^\circ, 270^\circ$.

Solution
We know that the angle $\theta = 0^\circ$ specifies the point $P = (1, 0)$. Since $\sin 0^\circ$ is the $y$-coordinate of this point and $\cos 0^\circ$ is the $x$-coordinate, this means $\sin 0^\circ = 0$ and $\cos 0^\circ = 1$. Likewise, we know that $\theta = 90^\circ$ specifies the point $P = (0, 1)$, so $\sin 90^\circ = 1$ and $\cos 90^\circ = 0$. Continuing, we can find the values of $\sin \theta$ and $\cos \theta$ for the other required angles as follows:

- At $\theta = 0^\circ$, $P = (1, 0)$ so $\cos 0^\circ = 1$ and $\sin 0^\circ = 0$
- At $\theta = 90^\circ$, $P = (0, 1)$ so $\cos 90^\circ = 0$ and $\sin 90^\circ = 1$
- At $\theta = 180^\circ$, $P = (-1, 0)$ so $\cos 180^\circ = -1$ and $\sin 180^\circ = 0$
- At $\theta = 270^\circ$, $P = (0,-1)$ so $\cos 270^\circ = 0$ and $\sin 270^\circ = -1$. 

The Sine and Cosine Functions In Right Triangles

If $\theta$ is an angle in a right triangle (other than the right angle),

$\sin \theta = \text{Opposite}/\text{Hypotenuse}$

$\cos \theta = \text{Adjacent}/\text{Hypotenuse}$

Consider the right triangle formed in the circle of radius $r$ in the Figure:

$\sin \theta = \frac{y}{r}$

$\cos \theta = \frac{x}{r}$
7.3

GRAPHS OF SINE AND COSINE
Tabulating and Graphing
Values of Sine and Cosine

Values of $\sin \theta$ and $\cos \theta$ for $0 \leq \theta < 360^\circ$

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$\sin \theta$</th>
<th>$\cos \theta$</th>
<th>$\theta$</th>
<th>$\sin \theta$</th>
<th>$\cos \theta$</th>
<th>$\theta$</th>
<th>$\sin \theta$</th>
<th>$\cos \theta$</th>
<th>$\theta$</th>
<th>$\sin \theta$</th>
<th>$\cos \theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>90</td>
<td>0</td>
<td>1</td>
<td>180</td>
<td>-1</td>
<td>0</td>
<td>270</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>30</td>
<td>0.87</td>
<td>0.5</td>
<td>120</td>
<td>-0.5</td>
<td>0.87</td>
<td>210</td>
<td>-0.87</td>
<td>-0.5</td>
<td>300</td>
<td>0.5</td>
<td>-0.87</td>
</tr>
<tr>
<td>45</td>
<td>0.71</td>
<td>0.71</td>
<td>135</td>
<td>-0.71</td>
<td>0.71</td>
<td>225</td>
<td>-0.71</td>
<td>-0.71</td>
<td>315</td>
<td>0.71</td>
<td>-0.71</td>
</tr>
<tr>
<td>60</td>
<td>0.5</td>
<td>0.87</td>
<td>150</td>
<td>-0.87</td>
<td>0.5</td>
<td>240</td>
<td>-0.5</td>
<td>-0.87</td>
<td>330</td>
<td>0.87</td>
<td>-0.5</td>
</tr>
</tbody>
</table>

$y = \cos \theta$

$y = \sin \theta$

$\theta$ in degrees

Functions Modeling Change:
A Preparation for Calculus,
Properties of Sine and Cosine

Properties of the sine and cosine functions that are apparent from the graph include:

• Domain: All values of $\theta$, since any angle, positive or negative, specifies a point on the unit circle.

• Range: Since values of the sine and cosine are coordinates of points on the unit circle, they lie between $-1$ and $1$. So the range of the sine and cosine are $-1 \leq \sin \theta \leq 1$ and $-1 \leq \cos \theta \leq 1$.

• Odd/Even Symmetry: The sine function is odd and the cosine function is even: $\sin(-\theta) = -\sin \theta$ and $\cos(-\theta) = \cos \theta$.

• Period: Both sine and cosine are periodic functions, because the values repeat regularly. The smallest interval over which the function values repeat—here $360^\circ$—is called the period. We have $\sin(\theta + 360^\circ) = \sin \theta$ and $\cos(\theta + 360^\circ) = \cos \theta$. 
Periodic Functions

A function $f$ is periodic if its values repeat at regular intervals. Then if the graph of $f$ is shifted horizontally by $c$ units, for some constant $c$, the new graph is identical to the original graph. In function notation, periodic means that, for all $t$ in the domain of $f$,

$$f(t + c) = f(t).$$

The smallest positive constant $c$ for which this relationship holds for all values of $t$ is called the period of $f$. 

Amplitude and Midline

Example 1

Compare the graph of \( y = \sin t \) to the graphs of \( y = 2 \sin t \) and \( y = -0.5 \sin t \), for \( 0^\circ \leq t \leq 360^\circ \). How are these graphs similar? How are they different? What are their amplitudes?

Solution

The graphs are shown. The amplitude of \( y = \sin t \) is 1, the amplitude of \( y = 2 \sin t \) is 2 and the amplitude of \( y = -0.5 \sin t \) is 0.5. The graph of \( y = -0.5 \sin t \) is “upside-down” relative to \( y = \sin t \). These observations are consistent with the fact that the constant \( A \) in the equation \( y = A \sin t \) stretches or shrinks the graph vertically, and reflects it about the \( t \)-axis if \( A \) is negative. The amplitude of the function is \(|A|\).
Midlines and Vertical Shifts

Example
Compare the graph of $y = \cos t$ to the graphs of $y = \cos t + 3$ and $y = \cos t - 2$, for $0^\circ \leq t \leq 360^\circ$. How are these graphs similar? How are they different? What are their midlines?

Solution
The graphs are shown. The midline of $y = \cos t$ is the $t$ axis ($y = 0$), the midline of $y = \cos t + 3$ is $y = 3$ and the midline of $y = \cos t - 2$ is $y = -2$. Recall that when the function $f(t)$ is shifted vertically by a distance $k$, the new function is $f(t) + k$. Similarly, the midline is shifted vertically by that same distance $k$. Generalizing, we conclude that the graphs of $y = \sin t + k$ and $y = \cos t + k$ have midlines $y = k$.

The $t$ axis is the midline for all three functions.
Coordinates of a Point on a Circle of Radius $r$

The coordinates of the point $P$ on the unit circle in the figure are given by $x = \cos \theta$ and $y = \sin \theta$.

The coordinates $(x, y)$ of the point $Q$ in the figure are given by $x = r \cos \theta$ and $y = r \sin \theta$. 

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Coordinates of Points on a Circle of Radius $r = 5$

**Example 2**

Find the coordinates of the points $A$, $B$, and $C$ in the Figure to three decimal places.

**Solution**

With $r = 5$, the coordinates of point $A$ are given by

- $x = 5 \cos 130^\circ = 5(-0.6427) = -3.214,$
- $y = 5 \sin 130^\circ = 5(0.766) = 3.830.$

Point $B$ corresponds to an angle of $-70^\circ$, (angle is measured clockwise), so $B$ has coordinates

- $x = 5 \cos(-70^\circ) = 5(0.342) = 1.710,$
- $y = 5 \sin(-70^\circ) = 5(-0.93969) = -4.698.$

For point $C$, we must first calculate the corresponding angle, since the $10^\circ$ is not measured from the positive x-axis. The angle we want is $180^\circ + 10^\circ = 190^\circ$, so

- $x = 5 \cos(190^\circ) = 5(-0.9848) = -4.924,$
- $y = 5 \sin(190^\circ) = 5(-0.1736) = -0.868.$
Height on the Ferris Wheel as a Function of Angle

Example 4
The Ferris wheel has a radius of 225 feet. Find your height above the ground as a function of the angle $\theta$ measured from the 3 o’clock position. What is your height when $\theta = 60^\circ$? when $\theta = 150^\circ$?

Solution
At $\theta = 60^\circ$, 
$ht = 225 + 225 \sin 60^\circ = 419.9$ ft.  
At $\theta = 150^\circ$,  
$ht = 225 + 225 \sin 150^\circ = 337.5$ ft.
Height on the Ferris Wheel as a Function of Angle

Example 5
Graph the Ferris wheel function giving your height, \( h = f(\theta) \), in feet, above ground as a function of the angle \( \theta \): \( f(\theta) = 225 + 225 \sin \theta \).
What are the period, midline, and amplitude?

Solution
A calculator gives the values to graph \( f(\theta) \). The period of this function is 360°, because 360° is one full rotation, so the function repeats every 360°. The midline is \( h = 225 \) feet, since the values of \( h \) oscillate about this value. The amplitude is also 225 feet, since the maximum value of \( h \) is 450 feet.

\[ f(\theta) = 225 + 225 \sin \theta \]

Period: 360°
Midline: \( y = 225 \)
Amplitude = 225 ft

\( h, \) height (feet)
7.4

THE TANGENT FUNCTION
Suppose \( P = (x, y) \) in the figure is the point on the unit circle specified by the angle \( \theta \). We define the function, tangent of \( \theta \), or \( \tan \theta \) by

\[
\tan \theta = \frac{y}{x} \text{ for } x \neq 0.
\]

Since \( x = \cos \theta \) and \( y = \sin \theta \), we see that

\[
\tan \theta = \frac{\sin \theta}{\cos \theta} \text{ for } \cos \theta \neq 0.
\]
The Tangent Function in Right Triangles

If $\theta$ is an angle in a right triangle (other than the right angle),

$$\tan \theta = \frac{a}{b} = \frac{\text{Opposite}}{\text{Adjacent}}$$
The Tangent Function in Right Triangles

Example 3
The grade of a road is calculated from its vertical rise per 100 feet. For instance, a road that rises 8 ft in every one hundred feet has a grade of 8%. 

Suppose a road climbs at an angle of 6° to the horizontal. What is its grade?

Solution
From the figure, we see that \( \tan 6° = \frac{x}{100} \), so, using a calculator, \( x = 100 \tan 6° = 10.510 \).
Thus, the road rises 10.51 ft every 100 feet, so its grade is 10.51/100 = 10.51%.
Interpreting the Tangent Function as Slope

We can think about the tangent function in terms of slope. In the Figure, the line passing from the origin through P has

\[
\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{y - 0}{x - 0} = \frac{y}{x} \quad \text{so} \quad \text{Slope} = \tan \theta.
\]

In words, \( \tan \theta \) is the slope of the line passing through the origin and point P.
Graphing the Tangent Function

• For values of $\theta$ between $180^\circ$ and $360^\circ$, observe that $\tan(\theta + 180^\circ) = \tan \theta$, because the angles $\theta$ and $\theta + 180^\circ$ determine the same line through the origin, and hence the same slope. Thus, $y = \tan \theta$ has period $180^\circ$.

• Since the tangent is not defined when the $x$-coordinate of $P$ is zero, the graph of the tangent function has a vertical asymptote at $\theta = -270^\circ, -90^\circ, 90^\circ, 270^\circ$, etc.

Graph of the tangent function

Theta (degrees)
7.5

RIGHT TRIANGLES:
INVERSE TRIGONOMETRIC FUNCTIONS
The Inverse Sine Function

For $0 \leq x \leq 1$: \[ \text{arcsin } x = \sin^{-1} x = \]

The angle in a right triangle whose sine is $x$.

Example 2
Use the inverse sine function to find the angles in the figure.

Solution
Using our calculator’s inverse sine function:

\[ \sin \theta = 3/5 = 0.6 \text{ so } \theta = \sin^{-1}(0.6) = 36.87^\circ \]
\[ \sin \phi = 4/5 = 0.8 \text{ so } \phi = \sin^{-1}(0.8) = 53.13^\circ \]

These values agree with the ones found in Example 1.
The Inverse Tangent Function

\[ \text{arctan } x = \tan^{-1} x = \text{The angle in a right triangle whose sine is } x. \]

Example 3
The grade of a road is 5.8%. What angle does the road make with the horizontal?

Solution
Since the grade is 5.8%, the road climbs 5.8 feet for 100 feet; see the figure. We see that
\[ \tan \theta = \frac{5.8}{100} = 0.058. \]
So
\[ \theta = \tan^{-1}(0.058) = 3.319^\circ \]
using a calculator.

A road rising at a grade of 5.8% (not to scale)
Summary of Inverse Trigonometric Functions

We define:
- the arc sine or inverse sine function as
  \[ \text{arcsin } x = \sin^{-1} x = \text{The angle in a right triangle whose sine is } x \]
- the arc cosine or inverse cosine function as
  \[ \text{arccos } x = \cos^{-1} x = \text{The angle in a right triangle whose cosine is } x \]
- the arc tangent or inverse tangent function as
  \[ \text{arctan } x = \tan^{-1} x = \text{The angle in a right triangle whose tangent is } x. \]

This means that for an angle \( \theta \) in a right triangle (other than the right angle),

\[
\begin{align*}
\sin \theta &= x \text{ means } \theta = \sin^{-1} x \\
\cos \theta &= x \text{ means } \theta = \cos^{-1} x \\
\tan \theta &= x \text{ means } \theta = \tan^{-1} x.
\end{align*}
\]
7.6

NON-RIGHT TRIANGLES
**The Law of Cosines**

**Law of Cosines**: For a triangle with sides $a$, $b$, $c$, and angle $C$ opposite side $c$, we have

$$c^2 = a^2 + b^2 - 2ab \cos C$$
Applying the Pythagorean theorem to the right-hand right triangle:

\[(a - x)^2 + h^2 = c^2 \text{ or } a^2 - 2ax + x^2 + h^2 = c^2.\]

Applying the Pythagorean theorem to the left-hand triangle, we get

\[x^2 + h^2 = b^2.\]

Substituting this result into the previous equation gives

\[a^2 - 2ax + (x^2 + h^2) = a^2 - 2ax + b^2 = c^2.\]

But \(\cos C = x/b\), so \(x = b \cos C\). This gives the Law of Cosines:

\[a^2 + b^2 - 2ab \cos C = c^2.\]
Application of the Law of Cosines

Example 1
A person leaves her home and walks 5 miles due east and then 3 miles northeast. How far has she walked? How far away from home is she?

Solution
She has walked $5 + 3 = 8$ miles in total.
One side of the triangle is 5 miles long, while the second side is 3 miles long and forms an angle of $135^\circ$ with the first.
This is because when the person turns northeast, she turns through an angle of $45^\circ$. Thus, we know two sides of this triangle, 5 and 3, and the angle between them, which is $135^\circ$. To find her distance from home, we find the third side $x$, using the Law of Cosines:

\[
x^2 = 5^2 + 3^2 - 2 \cdot 5 \cdot 3 \cos 135^\circ = 34 - 30 \left( -\frac{\sqrt{2}}{2} \right) = 55.213 , \quad x = \sqrt{55.213} \approx 7.431 \text{ miles}
\]

Notice that this is less than 8 miles, the total distance she walked.
The Law of Sines

**Law of Sines:** For a triangle with sides $a$, $b$, $c$ opposite angles $A$, $B$, $C$ respectively:

\[
\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}
\]
Proof of the Law of Sines

We derive the Law of Sines using the same triangle as in the proof of the Law of Cosines. Since

\[ \sin C = \frac{h}{b} \text{ and } \sin B = \frac{h}{c}, \]

we have \( h = b \sin C \) and \( h = c \sin B \).

This means that \( b \sin C = c \sin B \) and

\[ \frac{\sin B}{b} = \frac{\sin C}{c} \]

A similar type of argument (Problem 42) shows that

\[ \frac{\sin A}{a} = \frac{\sin B}{b}. \]
Application of the Law of Sines

Example 3
An aerial tram starts at a point one half mile from the base of a mountain whose face has a 60° angle of elevation. (See figure.) The tram ascends at an angle of 20°. What is the length of the cable from T to A?

Solution

The Law of Cosines does not help us here because we only know the length of one side of the triangle. We do however know two angles in this diagram and can determine the third. Thus, we can use the Law of Sines:

\[
\sin A/a = \sin C/c \text{ or } \sin 40°/0.5 = \sin 120°/c
\]

So

\[
c = 0.5 \left( \sin 120°/\sin 40° \right) = 0.674.
\]

Therefore, the cable from T to A is 0.674 miles.
When to Use the Laws of Cosines and Sines

• When two sides of a triangle and the angle between them are known the Law of Cosines is useful. It is also useful if all three sides of a triangle are known.
• The Law of Sines is useful when we know a side and the angle opposite it and one other angle or one other side.
• The Ambiguous Case: There is a drawback to using the Law of Sines for finding angles. The problem is that the Law of Sines does not tell us the angle, but only its sine, and there are two angles between 0° and 180° with a given sine. For example, if the sine of an angle is 1/2, the angle may be either 30° or 150°.